The Path Integral, Wilson Lines, and Disorder Operators

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Abstract

This lecture is intended to follow Phil's talk on "Categorical Geometric Langlands and Quantum Field Theory" and its relevance to the Langlands program. We start with a review of field theory, extending the ideas of classical field theory to the path integral formulation of quantum field theory (QFT). In particular we study the various physical "observables" that arise out of the path integral, introducing the operator product expansion and Wilson loops. Finally, we discuss *disorder operators*, whose insertion into the path integral changes the space of fields that we integrate over to include singularities. For this, we study 't Hooft lines via a the picture of monopoles in 3D.

Introduction

The "best hope" conjecture of Beilinson and Drinfeld for the geometric Langlands correspondence states

$$D(Bun_G(C)) \cong QC(Flat_{\check{G}}(C))$$

where $\check{G}$ is the Langlands dual group to G and everything else has been explained in reasonable depth by the lectures so far. Both sides of this equation have natural symmetries acting on them, and producing eigenvalue data that agrees. These are analogues of the *Hecke* and *Frobenius eigenvalues* of the arithmetic Langlands correspondence.

Goal 1. *Want to understand spectral decomposition of $D(Bun_G)$ and $QC(Flat_{\check{G}})$ using ideas from field theory.*

Goal 2. *To understand the symmetries that act naturally on both sides*

Before we can tackle these goals from the side of physics, we need to develop some understanding of the relevant objects that appear in the quantum field theory perspective.

1 Recall of Classical Field Theory and the Path Integral Formulation of QFT

Here is a slight reformulation of Phil's

Definition 1 (Classical Field Theory)**.** A classical field theory $\mathcal{E}$ is a collection of the following data:

- A manifold M known as the **spacetime** of the theory.

- A space of section maps $\Phi : M \to X$, where X is a Riemannian manifold called the "target space". Any such Φ is called a **field**.
- An action functional $S[\Phi]$ from the space of field configurations into $\mathbb{C}$ (or more generally some number field).

Classical field theory studies solutions to the **classical equations of motion**

$$\{\varphi \in \mathcal{F} \text{ s.t. } \delta S(\varphi) = 0\}.$$

Example 2. When $X = \mathbb{R}$, we get a single scalar field ϕ (here Φ is ϕ). An action for this field theory is often given by:

$$S[\phi] = \int_M |\partial_\mu \phi|^2.$$

Example 3. Classical electromagnetism is defined by $X = T^*M$ with an action given by:

$$S[A] = \int_M F \wedge \star F, \qquad F := \mathrm{d}A.$$

More generally, Yang-Mills theory is defined when $X = T^*M \otimes \mathfrak{g}$ and given

$$S[A] = \int_M \mathrm{Tr}\left(F \wedge \star F\right), \qquad F := \mathrm{d}A + A \wedge A.$$

where the trace is taken over the Lie algebra using the Killing form.

Though we do not know how to make sense of quantum field theory, the intuitive picture that we have of it is given by the **Feynman Path Integral**. For a given quantum field theory, there is quantity known as the **partition function**, defined as:

$$\mathcal{Z} = \int \mathcal{D}\Phi \, e^{-S[\Phi]}. \tag{1}$$

This is an integral taken over the space of all fields. The measure on this space is mathematically ill-defined in general.

Definition 4 (Classical Observable)**.** A classical observable is a functional from the set of field configurations to the ground field $\mathbf{k}$.

Definition 5 (Observable)**.** A **quantum observable** (which we will refer to as just an *observable* in these lectures) is a functional from the a field theory into the ground field $\mathbf{k}$. In the Feynman picture, it can be seen as a statistical average of classical observables over all field configurations.

The partition function is an observable, as is the **1-point correlation function** as x_1:

$$\langle \Phi(x_1) \rangle := \frac{1}{\mathcal{Z}} \int \mathcal{D}\Phi \, \Phi(x_1) e^{-S[\Phi]}.$$

In this example, the path integral over all configurations of Φ probes Φ at this single point, giving essentially an expectation value. We can take expectation values of many different operators, e.g. $\phi(x_1), \partial_\mu \phi(x_1), \mathbf{1}, \phi(x_1)\partial_\mu \phi(x_1)$ on X. We denote operators by $\mathcal{O}$. More generally, we define **correlation functions** as

$$\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle_g := \frac{1}{\mathcal{Z}} \int \mathcal{D}\Phi \, \mathcal{O}_1 \dots \mathcal{O}_n e^{-S[\Phi]}.$$

Definition 6 (TQFT)**.** If the correlation functions of a given quantum field theory are independent of the metric g, then the corresponding theory is called a **topological quantum field theory** (TQFT).

In fact metric independence implies diffeomorphism invariance.

Example 7 (Chern Simmons Theory)**.** It turns out the correlation functions of Chern-Simmons theory on a 3-manifold M with Φ being the field $A : M \to T^*M \otimes \mathfrak{g}$ and the action given by

$$S[A] \propto \int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$$

This is clear because the metric has no role in defining the action.

Proposition 8 (Operator Product Expansion)**.** *Within the path integral, a product of two local fields can be replaced by a (possibly infinite) sum over individual fields. Namely, given two operators* $\mathcal{O}_a, \mathcal{O}_b$ *and evaluation points* x_1, x_2*, there is an open neighborhood* U *around* x_2 *such that*

$$\mathcal{O}_a(x_1)\mathcal{O}_b(x_2) = \sum_c C^c_{ab}(x_1 - x_2)\mathcal{O}_c(x_2) \tag{2}$$

where $\mathcal{O}_c$ *are other operators in the quantum field theory, and the* C^c_{ab} *are analytic functions on* $U \backslash \{x_2\}$ *(that often become singular as* $x_1 \to x_2$*).*

In the 2D case, this yields the (possibly familiar) Laurent series associated with CFT. The structure constants contain valuable information about the QFT that allow onw to view it *algebraically*. In particular, they satisfy **associativity conditions**. The philosophy of the OPE is as follows **(elaborate Phil's point here)**. This leads naturally to the next idea

Idea 9. *The OPE coefficients, together with the 1-point correlation functions completely determine the* n*-point correlation functions.*

For example, a two-point function is simply given by:

$$\langle \mathcal{O}_a(x_1)\mathcal{O}_b(x_2)\rangle = \sum_c C^c_{ab}(x_1 - x_2)\langle \mathcal{O}_c(x_2)\rangle \tag{3}$$

2 Wilson Loops

In general, there are other observables in a quantum field theory beyond correlation functions. Consider a gauge theory:

Definition 10 (Gauge Theory)**.** A gauge theory is a field theory with the action invariant under the action of a Lie group G, known as the **gauge group** of the theory, acting at each point of M.

In general, gauge theories have a **connection 1-form**, denoted by A, which gives a way to transport data along any given vector bundle E associated to a representation R of G. This allows us to compare operators at different points

$$W_R(\gamma) = \exp\left(\int_\gamma A\right) \tag{4}$$

Wilson loops transform (under a general transformation $g \in \mathcal{G}$), as:

$$W_R(\gamma) = g(\gamma(1))W_R(\gamma)g(\gamma(0))^{-1} \tag{5}$$

in the special case of γ closed, we see this is gauge-invariant. This is called a **Wilson loop**. It can be viewed as yielding an element of the group G in the representation R. In this case, the trace of this element gives an invariant quantity, and so for γ closed we further add a trace.

Definition 11 (Wilson Loop)**.** Given a field theory with gauge group G and a finite-dimensional representation R of G together with a closed loop γ, we define the Wilson loop operator:

$$\mathcal{W}_R(\gamma) := \operatorname{Tr} R(\operatorname{Hol}(A, \gamma)). \tag{6}$$

The algebra of Wilson loops is simple. For $\gamma \to \gamma'$ the operator product expansion gives us that

$$W_R(\gamma)W_{R'}(\gamma') \in \operatorname{Span}_{R_i \subset R \otimes R'} W_{R_i}(\gamma). \tag{7}$$

3 Disorder Operators

We give the idea of **disorder operators** rather informally as:

Idea 12 (Disorder Operator)**.** *A disorder operator associated to a given submanifold γ of M constrains the fields in the path integral to have a certain singular behavior along γ.*

As an aside: from the Hilbert space perspective, we have so far been studying operator insertions between the vacuum state of the field theory:

$$\langle \mathcal{O}(x_1) \rangle = \langle 0 | \mathcal{O}(x_1) | 0 \rangle \,. \tag{8}$$

However, the vacuum state is only one of many possible classical solutions to the equations of motion for a given field theory. It is the one with zero action. In general, solutions to the equations of motion $\delta S = 0$ yielding a *finite, nonzero action* are called **instantons**.

Yang-Mills instantons on $\mathbb{R}^4$ are given by the **(anti-)self-dual** equations:

$$F = \pm \star F. \tag{9}$$

If we demand that A is translation-independent along the x_4 coordinate and define $\phi = A_4$, these equations become the **Bogomolny equations** for magnetic monopoles on $\mathbb{R}^3$[1]

$$F = \star D_A \phi, \qquad D_A := \mathrm{d} + A. \tag{10}$$

Consider the sphere S^2_R of radius R in $\mathbb{R}^3$. Given a solution of the Bogomolny equations for A, the associated curvature form F will give a certain Chern class to any vector bundle E over the sphere.

Ordinarily, you would argue "$\mathbb{R}^4$ is trivial, the only vector bundles living on this should be globally trivializable". However, the solutions of the Bogomolny equations *induce singularities* in the vector bundle structure. These singularities effectively change the topology of $\mathbb{R}^3$ so that nonzero Chern classes can exist for the bundles over the space. In this way, insertions of monopoles act as operators, changing the vector bundles of the fields we integrate over.

[1]Note that these solutions are not instantons, as translation invariance along x_4 does not allow these classical solutions to have the necessary decay conditions to yield finite action on $\mathbb{R}^4$. Still, when viewed on $\mathbb{R}^3$, it can be shown that these solutions *do* give a finite action indexed by an integer known as the *monopole number*.

Definition 13 ('t Hooft Operator)**.** Given a closed curve γ and an element $\mu \in \mathfrak{g}$ defined up to adjoint action, the 't Hooft operator $T_\mu(\gamma)$ localized on γ corresponding to μ demands that for any point p on γ, in local coordinates $x_1 = x_2 = x_3 = 0$ defining γ we have

$$F(x) \sim \star_3 d\left(\frac{\mu}{2r}\right) \tag{11}$$

with $\star_3$ the hodge star in the codimension 1 hyperplane perpendicular to γ.

More on this correspondence in lecture 2